THE INTEGRAL CALCULUS APPLIED TO PLANE CURVES.

SUCCESSIVE INTEGRATION.

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PAGE

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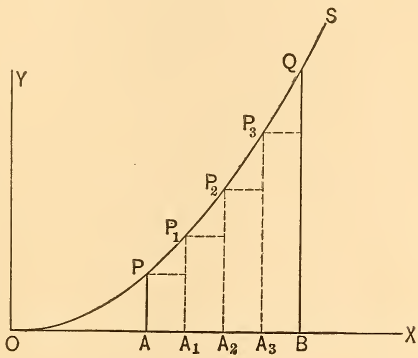
XXII. INTEGRATION AS A SUMMATION. DEFINITE INTEGRALS.

201. The most common application of integration is the summation of an infinite series of infinitely small terms. As an illustration, consider the following problem.



202. To find the area $PABQ$ included between a given curve OS, the axis of X, and the ordinates AP and BQ.

Let $y = x^3$ be the equation of the given curve. Let $OA = a$, $OB = b$. Suppose AB divided into n equal parts (in the figure, $n = 4$), and let Δx denote one of the equal parts, as AA_1, A_1A_2, $\cdots$.

Then $$AB = b - a = n\Delta x.$$

At A_1, A_2, $\cdots$, draw the ordinates A_1P_1, A_2P_2, $\cdots$, and complete the rectangles PA_1, P_1A_2, $\cdots$.

From the equation of the curve, $y = x^3$,

$$PA = a^3,\ P_1A_1 = (a + \Delta x)^3,\ P_2A_2 = (a + 2\,\Delta x)^3,\ \cdots\ QB = b^3.$$

Area of rectangle $PA_1 \ = PA \ \times AA_1 \ = a^3\Delta x$.

Area of rectangle $P_1A_2 = P_1A_1 \times A_1A_2 = (a + \Delta x)^3\Delta x$.

Area of rectangle $P_2A_3 = P_2A_2 \times A_2A_3 = (a + 2\,\Delta x)^3\Delta x$.

. . . .

The sum of all the n rectangles is

$$a^3\Delta x + (a + \Delta x)^3\Delta x + (a + 2\,\Delta x)^3\,\Delta x + \cdots + (b - \Delta x)^3\,\Delta x,$$

which may be represented by the symbol $\sum_a^b x^3\Delta x$, for each term of the series is represented by $x^3\Delta x$, x taking in succession the values a, $a+\Delta x$, $a+2\Delta x$, $\cdots$, $b-\Delta x$. Thus,

$$a^3\Delta x + (a+\Delta x)^3\Delta x + (a+2\,\Delta x)^3\Delta x + \cdots$$
$$+ (b-\Delta x)^3\Delta x = \sum_a^b x^3\Delta x.$$

203. It is evident that the area $PABQ$ is the limit of the sum of the rectangles, as n approaches infinity. Δx at the same time approaches the infinitesimal dx, and instead of $\sum_a^b x^3\Delta x$, we write $\int_a^b x^3dx$. Thus, making $n=\infty$, we have

$$PABQ = a^3dx + (a+dx)^3dx + (a+2\,dx)^3dx + \cdots$$
$$+ (b-dx)^3dx = \int_a^b x^3dx \; . \quad . \quad . \quad . \quad . \qquad (a)$$

The symbol $\int_a^b x^3dx$ as defined by (a) denotes the sum of an infinite number of terms, each of which is represented by x^3dx, x taking in succession the values a, $a+dx$, $a+2\,dx$, $\cdots$ $b-dx$.

It is to be noticed that a new definition is thus given to the symbol $\int$, a definition which will subsequently appear to be perfectly consistent with that hitherto assumed, where it denotes the inverse of differentiation.

204. To find the area $PABQ$, we must find the sum of the series (a), that is, the value of $\int_a^b x^3dx$.

Now $$\int x^3dx = \frac{x^4}{4};$$

that is, $$x^3dx = d\left(\frac{x^4}{4}\right) = \frac{(x+dx)^4}{4} - \frac{x^4}{4}. \quad . \quad . \quad . \qquad \text{(Art. 5.)}$$

Substituting in this equation for x,

$$a,\; a+dx,\; a+2\,dx,\; \cdots\; b-dx,$$

we have
$$a^3dx = \frac{(a+dx)^4}{4} - \frac{a^4}{4},$$
$$(a+dx)^3dx = \frac{(a+2\,dx)^4}{4} - \frac{(a+dx)^4}{4},$$
$$(a+2\,dx)^3dx = \frac{(a+3\,dx)^4}{4} - \frac{(a+2\,dx)^4}{4},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(b-dx)^3dx = \frac{b^4}{4} - \frac{(b-dx)^4}{4}.$$

Adding and cancelling terms in second member, we have
$$a^3dx + (a+dx)^3dx + (a+2\,dx)^3dx + \cdots + (b-dx)^3dx = \frac{b^4}{4} - \frac{a^4}{4}.$$

Or
$$\int_a^b x^3dx = \frac{b^4}{4} - \frac{a^4}{4} = \text{area } PABQ.$$

We have thus shown that the sum of the infinite series represented by $\int_a^b x^3dx$ is found by substituting b and a in succession in $\frac{x^4}{4}$, and subtracting the latter result from the former, — the function $\frac{x^4}{4}$ being the integral of x^3dx, using the word *integral* in the sense heretofore used.

205. The relation of the terms of the series (α), Art. 203, to the integral $\frac{x^4}{4}$ may be made clearer to the student by considering the following series of numbers: —

$$\begin{array}{cc} 1 & \\ & 3 \\ 4 & \\ & 5 \\ 9 & \\ & 7 \\ 16 & \\ & 9 \\ 25 & \\ & 11 \\ 36 & \end{array}$$

The numbers in the second column are the differences between consecutive numbers in the first, and it is evident that the sum of the second column of numbers equals the difference between the first and last in the first column. That is,

$$3 + 5 + 7 + 9 + 11 = 36 - 1.$$

The terms of the series (a), Art. 203, may be similarly arranged, as follows: —

$$\begin{array}{ll}
\dfrac{a^4}{4} & \\
 & a^3dx \\
\dfrac{(a+dx)^4}{4} & \\
 & (a+dx)^3dx \\
\dfrac{(a+2\,dx)^4}{4} & \\
 & (a+2\,dx)^3dx \\
\dfrac{(a+3\,dx)^4}{4} & \\
\cdots\cdots & \cdots\cdots \\
\dfrac{(b-dx)^4}{4} & \\
 & (b-dx)^3dx \\
\dfrac{b^4}{4} &
\end{array}$$

Since x^3dx is the differential of $\frac{x^4}{4}$, the terms in the second column are the infinitesimal differences between the consecutive terms in the first, and therefore

$$a^3dx + (a+dx)^3dx + \cdots + (b-dx)^3dx = \frac{b^4}{4} - \frac{a^4}{4};$$

that is,

$$\int_a^b x^3dx = \frac{b^4}{4} - \frac{a^4}{4}.$$

206. The expression $\int_a^b x^3dx$ in Arts. 203, 204, is called a *definite integral*, and the process of evaluating it is called *integrating between limits.* b is the *superior*, and a the *inferior*, limit.

In contradistinction, $\frac{x^4}{4}$ is called the *indefinite integral* of x^3dx.

In general, if $\phi(x)$ denote any given function of x, $\int_a^b \phi(x)\,dx$ is the definite integral representing an infinite series of terms obtained from $\phi(x)\,dx$ by supposing x to vary from a to b. If $\psi(x) = \int \phi(x)\,dx$, the indefinite integral,

then
$$\int_a^b \phi(x)\,dx = \psi(b) - \psi(a).$$

It is to be noticed that the arbitrary constant in the indefinite integral disappears from the definite integral.

Thus, if in evaluating $\int_a^b x^3dx$, we call the indefinite integral $\frac{x^4}{4} + c$, we have

$$\int_a^b x^3dx = \frac{b^4}{4} + c - \left(\frac{a^4}{4} + c\right) = \frac{b^4}{4} - \frac{a^4}{4}, \text{ as before.}$$

EXAMPLES.

Evaluate the following definite integrals : —

1. $\int_1^4 x^2dx = \frac{x^3}{3}\Big|_1^4 = \frac{64}{3} - \frac{1}{3} = 21.$

2. $\int_1^e \frac{dx}{x} = \log x\Big|_1^e = \log e - \log 1 = 1.$

3. $\int_0^{\frac{\pi}{2}} \sin x\,dx = -\cos x\Big|_0^{\frac{\pi}{2}} = 0 - (-1) = 1.$

4. $\int_0^b (b^2x - x^3)\,dx = \frac{b^4}{4}.$

5. $\int_1^4 \frac{dx}{x^{\frac{3}{2}}} = 1.$

6. $\int_2^3 \frac{x\,dx}{1 + x^2} = \frac{\log 2}{2}.$

7. $\int_0^\infty \frac{8\,a^3dx}{x^2 + 4\,a^2} = 2\,\pi a^2.$

8. $\int_0^{\frac{\pi}{4}} \sec^4\theta\,d\theta = \frac{4}{3}.$

9. $\int_1^e x\log x\,dx = \frac{e^2 + 1}{4}.$

10. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{dx}{\cos x} = \log\left(\frac{1 + \sqrt{2}}{\sqrt{3}}\right).$

11. $\int_1^\infty \frac{dx}{x^2 - 2x\cos\alpha + 1} = \frac{\pi - \alpha}{2\sin\alpha}.$

207. *Change of Limits.* Where a new variable is used in obtaining the indefinite integral, it is well to make a corresponding change in the limits.

For example, to evaluate $\int_0^4 \frac{dx}{1+\sqrt{x}}$, put $\sqrt{x} = z$.

Then we have $$\frac{dx}{1+\sqrt{x}} = \frac{2\,z\,dz}{1+z}.$$

Since $z = 2$ when $x = 4$, and $z = 0$ when $x = 0$,

$$\therefore \int_0^4 \frac{dx}{1+\sqrt{x}} = \int_0^2 \frac{2\,z\,dz}{1+z} = 2[z - \log(1+z)]\Big|_0^2 = 4 - 2\log 3.$$

EXAMPLES.

1. $\int_4^5 \frac{3x-1}{(x-3)^2}dx = 4 + 3\log 2.$ Put $x - 3 = z$.

2. $\int_0^{e-1} x\log(x+1)dx = \frac{e^2-3}{4}.$ Put $x + 1 = z$.

3. $\int_1^e x^3(\log x)^2 dx = \frac{5e^4-1}{32}.$ Put $\log x = z$.

4. $\int_0^{\frac{\pi}{2}} \frac{d\theta}{4+5\sin\theta} = \frac{\log 2}{3}.$

Put $\sin\theta = x$; afterwards $\sqrt{1-x^2} = (1+x)z$.

XXIII. APPLICATION OF THE INTEGRAL CALCULUS TO PLANE CURVES.

208. *Quadrature of Areas. Rectangular Coördinates.* To find the area included within a given curve, the axis of X, and two given ordinates.

We have already given the solution of this problem in Arts. 202–204, as an illustration of a definite integral.

209. We may also regard the required area as generated by the ordinate AP (see Fig., Art. 202) moving from left to right, and varying in length according to the equation of the given curve. Regarding y as constant while moving the distance dx, it generates the rectangle ydx. Then the general formula for the required area is

$$A = \int_a^b y\,dx,$$

the inferior limit $a = OA$ denoting the initial position of the moving ordinate, and the superior limit $b = OB$, its final position.

EXAMPLES.

1. Find the area between the parabola $y^2 = 4ax$ and the axis of X, from the origin to the ordinate at the point (h, k).

Here $$A = \int^h y\,dx = \int_0^h 2a^{\frac{1}{2}}x^{\frac{1}{2}}\,dx = \frac{4a^{\frac{1}{2}}x^{\frac{3}{2}}}{3}\Big|_0^h = \frac{4a^{\frac{1}{2}}h^{\frac{3}{2}}}{3}.$$

Since $k^2 = 4ah,\ k = 2a^{\frac{1}{2}}h^{\frac{1}{2}}$,

$\therefore\ A = \frac{2}{3}hk$, two-thirds the circumscribed rectangle.

2. Find the entire area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. *Ans.* πab.

3. Find the area between the equilateral hyperbola $2xy = a^2$ and the axis of X, from $x = h$ to $x = 4h$. *Ans.* $a^2 \log 2$.

4. Find the entire area between the witch $y = \dfrac{8a}{x^2 + 4a^2}$ and the axis of X.

Ans. $4\pi a^2$.

5. Find the area intercepted between the coördinate axes by the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

Ans. $\frac{a^2}{6}$.

6. Find the entire area within the curve $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

Ans. $\frac{3}{4}\pi ab$.

7. Find the entire area within the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ans. $\frac{3\pi a^2}{8}$.

8. Find the entire area between the cissoid $y^2 = \frac{x^3}{2a - x}$ and the line $x = 2a$, its asymptote.

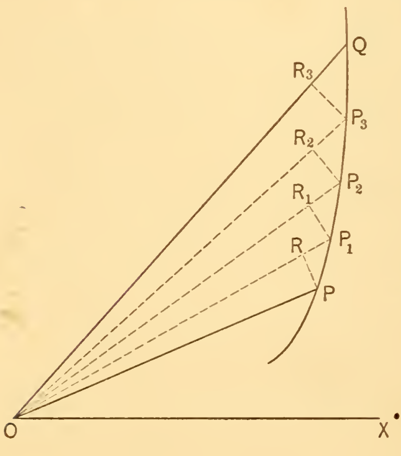
Ans. $3\pi a^2$.

The area between two curves is the sum, or the difference, of the areas between the curves and one of the coördinate axes, the limits being determined by the points of intersection.

9. Find the area included between the parabola $x^2 = 4ay$ and the witch $y = \frac{8a^3}{x^2 + 4a^2}$.

Ans. $(2\pi - \frac{4}{3})\, a^2$.

210. *Polar Coördinates.* To find the area POQ included within a given curve and two given radii vectores, OP and OQ. Let



$$POX = \alpha, \quad QOX = \beta.$$

Let r and θ be the coördinates of any point P_2 of the curve, then

$$r + \Delta r, \quad \theta + \Delta\theta,$$

will be the coördinates of P_3.

The area of the circular sector P_2OR_2 is

$$\tfrac{1}{2} OP_2 \times P_2R_2 = \tfrac{1}{2} r \cdot r\Delta\theta$$
$$= \tfrac{1}{2} r^2 \Delta\theta.$$

The sum of the sectors POR, P_1OR_1, P_2OR_2, $\cdots$ would be $\Sigma \frac{1}{2} r^2 \Delta\theta$.

The required area POQ is the limit of the sum of the sectors as $\Delta\theta$ approaches zero. That is

$$A = \tfrac{1}{2}\int_\alpha^\beta r^2 d\theta.$$

211. We may also regard the area POQ as generated by the radius vector revolving from OP to OQ, and varying in length according to the equation of the given curve PQ.

Regarding r as constant while describing the angle $d\theta$, it generates the sector whose area is $\frac{1}{2} r^2 d\theta$.

$$\therefore\ A = \tfrac{1}{2}\int_\alpha^\beta r^2 d\theta, \text{ as before;}$$

the inferior limit α denoting the initial, and the superior limit β, the final position, of the moving radius vector.

EXAMPLES.

1. Find the area described by the radius vector in one entire revolution of the spiral of Archimedes $r = a\theta$.

Here $$A = \tfrac{1}{2}\int_0^{2\pi} r^2 d\theta = \tfrac{1}{2}\int_0^{2\pi} a^2\theta^2 d\theta = \frac{a^2}{2}\frac{\theta^3}{3}\Big|_0^{2\pi} = \frac{4\pi^3 a^2}{3}.$$

2. Find the area described by the radius vector in the logarithmic spiral $r = e^{a\theta}$, from $\theta = 0$ to $\theta = \frac{\pi}{2}$. *Ans.* $\frac{1}{4a}(e^{\pi a} - 1)$.

3. Find the entire area of the circle $r = a \sin\theta$.

Ans. $\frac{\pi a^2}{4}$.

4. Find the area of one loop of the curve $r = a \sin 2\theta$.

Ans. $\frac{\pi a^2}{8}$.

5. Find the entire area of the cardioid $r = a(1 - \cos\theta)$.

Ans. $\frac{3\pi a^2}{2}$, or six times the area of the generating circle.

6. Find the area described by the radius vector in the parabola $r = a\sec^2\frac{\theta}{2}$, from $\theta = 0$ to $\theta = \frac{\pi}{2}$. *Ans.* $\frac{4a^2}{3}$.

7. Find the area below OX within the curve $r = a\sin^3\frac{\theta}{3}$.

Ans. $(10\pi + 27\sqrt{3})\frac{a^2}{64}$.

212. *Rectification of Curves. Rectangular Coördinates.* To find the length of the arc of a curve between two given points.

Denoting this length by s, we have from (38), Art. 87,

$$ds = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx,$$

therefore
$$s = \int \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx,$$

the limits being the limiting values of x.

Or we may evidently use the formula

$$s = \int \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}} dy,$$

the limits being the limiting values of y.

EXAMPLES.

1. Find the length of the arc of the parabola $y^2 = 4ax$, from the vertex to the extremity of the latus rectum.

Here
$$\frac{dy}{dx} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}},$$

therefore
$$s = \int_0^a \left(1 + \frac{a}{x}\right)^{\frac{1}{2}} dx = \int_0^a \left(\frac{a+x}{x}\right)^{\frac{1}{2}} dx.$$

To integrate, put $\sqrt{x} = z$, then

$$\int_0^a \left(\frac{a+x}{x}\right)^{\frac{1}{2}} dx = 2\int_0^{\sqrt{a}} \sqrt{a + z^2}\, dz$$
$$= z\sqrt{a+z^2} + a\log\left(z + \sqrt{a+z^2}\right)\Big|_0^{\sqrt{a}}.$$

$\therefore\ s = a\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right] = 2.29558\,a.$

2. Find the length of the arc of the semi-cubical parabola $ay^2 = x^3$, from the origin to $x = 5a$. *Ans.* $\frac{335a}{27}$.

3. Find the length of the arc of the curve $9ay^2 = x(x-3a)^2$, from $x = 0$ to $x = 3a$. *Ans.* $2a\sqrt{3}$.

4. Find the length of the arc of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, from $x = 0$ to the point (x, y).

Ans. $\frac{a}{2}(e^{\frac{x}{a}} - e^{-\frac{x}{a}})$.

5. Find the entire length of the arc of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $6a$.

213. *Polar Coördinates.* To find the length of the arc of a curve between two given points.

We have from (40), Art. 87,

$$ds = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta,$$

therefore
$$s = \int \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta, \quad . \quad . \quad . \quad . \quad (a)$$

the limits being the limiting values of θ.

Or we have also from Art. 87,

$$ds = \left[1 + r^2\left(\frac{d\theta}{dr}\right)^2\right]^{\frac{1}{2}} dr.$$

therefore
$$s = \int \left[1 + r^2\left(\frac{d\theta}{dr}\right)^2\right]^{\frac{1}{2}} dr, \quad . \quad . \quad . \quad . \quad (b)$$

the limits being the limiting values of r.

EXAMPLES.

1. Find the length of the arc of the spiral of Archimedes $r = a\theta$, from the pole to the end of the first revolution.

Here
$$\frac{dr}{d\theta} = a.$$

$$s = \int_0^{2\pi} (a^2\theta^2 + a^2)^{\frac{1}{2}} d\theta = a\int_0^{2\pi} (1 + \theta^2)^{\frac{1}{2}} d\theta$$

$$= a\left[\frac{\theta\sqrt{1+\theta^2}}{2} + \tfrac{1}{2}\log(\theta + \sqrt{1+\theta^2})\right]_0^{2\pi}$$

$$= a[\pi\sqrt{1+4\pi^2} + \tfrac{1}{2}\log(2\pi + \sqrt{1+4\pi^2})].$$

2. Find the entire length of the cardioid $r = a(1 - \cos\theta)$.

Ans. $8a$.

3. Find the length of the logarithmic spiral $r = e^{a\theta}$, from the pole to the point (r, θ). Use Formula (b).

Ans. $\frac{r}{a}\sqrt{a^2+1}$.

4. Find the entire length of the curve $r = a\sin^3\frac{\theta}{3}$.

Ans. $\frac{3\pi a}{2}$.

5. The equation of the epicycloid, the radius of the fixed circle being a, and that of the rolling circle $\frac{a}{2}$, is

$$\sin^2\theta = \frac{4(r^2 - a^2)^3}{27a^4r^2}.$$ Find the length of one loop.

From the above equation

$$\frac{d\theta}{dr} = \frac{2\sqrt{r^2-a^2}}{r\sqrt{4a^2-r^2}};$$ then use Formula (b).

Ans. $6a$.

214. *Surfaces of Revolution.* To find the *volume* of the surface of revolution generated by revolving a given plane curve about one of the coördinate axes.

Suppose X to be the axis of revolution. Then we may regard the required volume as generated by the area of a circle, which moves with its plane always perpendicular to the axis, its centre moving along this axis, and its radius being the ordinate of the given curve.

Since y is the radius of this moving circle, its area is πy^2,

and regarding y as constant while it moves over the distance dx, we have for the volume of an elementary cylinder

$$dV = \pi y^2 dx.$$

$$\therefore\ V = \pi \int y^2 dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

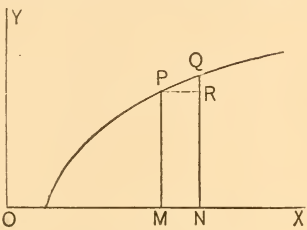
the limits being the limiting values of x.

Similarly, if Y is the axis of revolution,

$$V = \pi \int x^2 dy,$$

the limits being the limiting values of y.

215. To find the *surface* generated by revolving a given plane curve about one of the coördinate axes.



Suppose X the axis of revolution.

Let PQ be an element of the given curve. This will generate the convex surface of the frustum of a cone.

Hence we have for an element of the required surface

$$dS = 2\pi \left(\frac{PM + QN}{2}\right) PQ$$

$$= \pi\,(y + y + dy)\,ds$$

$$= 2\pi y ds + \pi dy ds.$$

Omitting the last term as infinitely small in comparison with the first, we have

$$dS = 2\pi y ds.$$

$$\therefore\ S = 2\pi \int y ds;$$

by (38), Art. 87, $$S = 2\pi \int y \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx \quad . \quad . \quad . \quad (a)$$

Similarly if Y is the axis of revolution,

$$S = 2\pi \int x ds.$$

EXAMPLES.

1. Find the volume and surface of the prolate spheroid obtained by revolving about X the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

From (a) Art. 214, we have

$$\tfrac{1}{2}V = \pi\int_0^a y^2dx = \pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx = \frac{2\pi ab^2}{3}.$$

$$\therefore\quad V = \frac{4\pi ab^2}{3}.$$

From (a), Art. 215,

$$\tfrac{1}{2}S = 2\pi\int_0^a y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx$$

$$= 2\pi\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\left[1+\frac{b^2x^2}{a^2(a^2-x^2)}\right]^{\frac{1}{2}}dx$$

$$= 2\pi\frac{b}{a^2}\int_0^a [a^4+(a^2-b^2)x^2]^{\frac{1}{2}}\,dx$$

$$= \pi b\left(b+\frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right).$$

$$\therefore\quad S = 2\pi b\left(b+\frac{a^2}{\sqrt{a^2-b^2}}\cos^{-1}\frac{b}{a}\right).$$

2. Find the volume and surface generated by revolving about X the parabola $y^2 = 4ax$, from the origin to $x = a$.

Ans. $2\pi a^3$ and $\frac{8(\sqrt{8}-1)}{3}\pi a^2$.

3. Find the volume and convex surface of the right cone generated by revolving about X the line joining the origin and the point (a, b). *Ans.* $\frac{\pi ab^2}{3}$ and $\pi b\sqrt{a^2+b^2}$.

4. Find the entire volume and surface generated by revolving about X the hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

Ans. $\frac{32\pi a^3}{105}$ and $\frac{12\pi a^2}{5}$.

5. Find the entire volume generated by revolving the witch $y = \frac{8a^3}{x^2 + 4a^2}$ about X, its asymptote. *Ans.* $4\pi^2 a^3$.

6. Find the volume generated by revolving about X the part of the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ intercepted by the coördinate axes.

Ans. $\frac{\pi a^3}{15}$.

7. Find the volume and surface of the torus obtained by revolving about X the circle $x^2 + y^2 - 2ay = 0$.

Ans. $2\pi^2 a^3$ and $4\pi^2 a^2$.

8. Find the volume and surface generated by revolving about Y the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ from $x = 0$ to $x = a$.

Ans. $\frac{\pi a^3}{2}(e + 5e^{-1} - 4)$ and $2\pi a^2(1 - e^{-1})$.

XXIV. SUCCESSIVE INTEGRATION.

216. *Double Integral.* If we reverse the operations represented by $\frac{d^2u}{dy\,dx}$ we have what is called a *double integral.*

For example, suppose $\frac{d^2u}{dy\,dx} = x^2y^3,$

then
$$u = \int\int x^2y^3dy\,dx,$$

which indicates two successive integrations, the first with reference to x regarding y as a constant, and the second with reference to y regarding x as a constant. Thus

$$u = \int \frac{x^3y^3}{3}dy = \frac{x^3y^4}{12},$$

omitting the constants of integration.

217. *Definite Double Integral.* Here the integrations are between given limits.

For example

$$\int_b^{2b}\int^a (a-x)\,y^2dy\,dx = \int_b^{2b}\left(ax - \frac{x^2}{2}\right)_0^a y^2dy$$
$$= \int_b^{2b}\frac{a^2}{2}y^2dy = \frac{7\,a^2b^3}{6}.$$

In the above $\int_b^{2b}\int_0^a (a-x)\,y^2dy\,dx$, the right integral sign with the limits 0 and a, is to be used with the variable x, and the left with the limits b and $2\,b$, with the variable y; that is, the integral signs with their limits are to be taken in the same order as the differentials dy, dx, at the end, and from *right* to *left.*

218. Sometimes the limits of the first integration are functions of the variable of the second.

For example,

$$\int_0^a\int_{y-a}^{2y} xy\,dy\,dx = \int_0^a \left(\frac{x^2}{2}\right)_{y-a}^{2y} y\,dy = \tfrac{1}{2}\int_0^a (3y^3 + 2ay^2 - a^2y)\,dy$$

$$= \frac{11a^4}{24}.$$

As another example,

$$\int_0^a\int_0^{\sqrt{a^2-x^2}} (x+y)\,dx\,dy = \int_0^a \left(xy + \frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dy$$

$$= \int_0^a \left(x\sqrt{a^2-x^2} + \frac{a^2-x^2}{2}\right) dx = \frac{2a^3}{3}.$$

219. *Triple Integrals.* A similar notation is used for three successive integrations. Thus

$$\int_b^a\int_0^b\int_a^{2a} x^2y^2z\,dx\,dy\,dz = \int_b^a\int_0^b \frac{3a^2}{2}x^2y^2\,dx\,dy$$

$$= \frac{3a^2}{2}\int_b^a \frac{b^3}{3}x^2dx = \frac{a^2b^3}{2}\left(\frac{a^3}{3} - \frac{b^3}{3}\right) = \frac{a^2b^3}{6}(a^3 - b^3).$$

EXAMPLES.

Evaluate the following definite integrals :—

1. $\displaystyle\int_0^a\int_0^b xy(x-y)\,dx\,dy = \frac{a^2b^2}{6}(a-b).$

2. $\displaystyle\int_b^a\int_\beta^a r^2\sin\theta\,dr\,d\theta = \frac{a^3-b^3}{3}(\cos\beta - \cos a).$

3. $\displaystyle\int_0^{2a}\int_{\frac{y^2}{4a}}^{3a-y} (x^2+y^2)\,dy\,dx = \frac{314a^4}{35}.$

4. $\displaystyle\int_{\frac{b}{2}}^b\int_0^{\frac{r}{b}} r\,dr\,d\theta = \frac{7b^2}{24}.$

5. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{a\sec^2\frac{1}{2}\theta} r^3\,d\theta\,dr = \frac{48a^4}{35}.$

6. $\int_a^{2a}\int_0^x\int_y^x xyz\,dx\,dy\,dz = \frac{4a^6}{3}.$

7. $\int_0^1\int_0^x\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz = \frac{e^4-3}{8} - \frac{3e^2}{4} + e.$

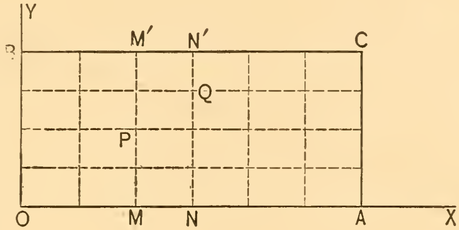
XXV. DOUBLE INTEGRATION APPLIED TO PLANE AREAS AND MOMENT OF INERTIA.

220. Certain problems connected with a plane area require a double integral. As an illustration, we shall consider the problem of finding the *moment of inertia* of a given plane area.

Definition. The *moment of inertia* of a given plane area about a given point in the plane, is the sum of the products obtained by multiplying the area of each infinitesimal portion by the square of its distance from the given point.

221. To find the moment of inertia of the rectangle $OACB$ about O.

Let $OA = a$,
$OB = b$.



Suppose the rectangle divided into rectangular elements by lines parallel to the coördinate axes. Let x, y, which are to be regarded as independent variables, be the coördinates of any point of intersection as P, and $x + dx$, $y + dy$, the coördinates of Q. Then the area of the element PQ is $dx\,dy$.

$$\text{Moment of } PQ = \overline{OP}^2 \cdot dx\,dy = (x^2 + y^2)\,dx\,dy.$$

The moment of the entire rectangle $OACB$ is the sum of all the terms obtained from $(x^2 + y^2)\,dx\,dy$ by varying x from 0 to a, and y from 0 to b.

If we suppose x to be constant while y varies from 0 to b, we shall have the terms that constitute a vertical strip $MNN'M'$.

Hence,

$$\text{Moment of } MNN'M' = dx\int_0^b (x^2+y^2)\,dy$$

$$= dx\left(x^2y+\frac{y^3}{3}\right)_0^b = \left(bx^2+\frac{b^3}{3}\right)dx.$$

Having thus found the moment of a vertical strip, we may sum all those strips by supposing x in this result to vary from 0 to a. That is,

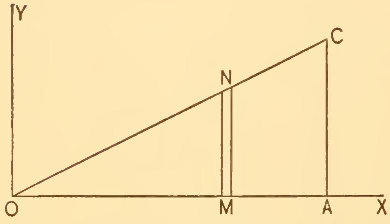
$$\text{Moment of } OACB = \int_0^a \left(bx^2+\frac{b^3}{3}\right)dx = \frac{a^3b+ab^3}{3}.$$

But the preceding operations are the same as those represented by the double integral

$$\int_0^a\int_0^b (x^2+y^2)\,dx\,dy. \quad \text{See Art. 217.}$$

If we first collect all the elements in a *horizontal* strip, and then sum these horizontal strips, we have

$$\text{Moment of } OACB = \int_0^b\int_0^a (x^2+y^2)\,dy\,dx = \frac{a^3b+ab^3}{3}.$$



222. To find the moment of inertia of the right triangle OAC about O.

Let $OA=a$, $AC=b$.

The equation of OC is

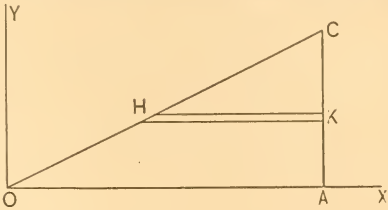
$$y=\frac{b}{a}x.$$

This differs from the preceding problem only in the limits of the first integration. In collecting the elements in a vertical strip MN, y varies from 0 to MN. But MN is no longer a constant as in Art. 221, but varies with OM, according to the equation of OC, $y=\dfrac{b}{a}x$. Hence the limits of y are 0 and $\dfrac{b}{a}x$.

In collecting all the vertical strips by the second integration, x varies from 0 to a as in Art. 221.

$$\therefore \text{ Moment of } OAC = \int_0^a \int_0^{\frac{bx}{a}} (x^2+y^2)\,dx\,dy = ab\left(\frac{a^2}{4}+\frac{b^2}{12}\right).$$

By supposing the triangle composed of *horizontal* strips as HK, we shall find



Moment of OAC

$$= \int_0^b \int_{\frac{ay}{b}}^a (x^2+y^2)\,dy\,dx$$

$$= ab\left(\frac{a^2}{4}+\frac{b^2}{12}\right).$$

223. *Plane Area as a Double Integral.* If in Art. 221 we omit the factor (x^2+y^2), we shall have instead of the moment, the area of the given surface.

That is, $$\text{Area} = \int\!\!\int dx\,dy = \int\!\!\int dy\,dx,$$

the limits being determined as before.

EXAMPLES.

1. Find the moment of inertia about the origin, of the right triangle formed by the coördinate axes and the line joining the points $(a, 0)$, $(0, b)$.

$$\textit{Ans.} \int_0^a \int_0^{\frac{b(a-x)}{a}} (x^2+y^2)\,dx\,dy = \frac{ab(a^2+b^2)}{12}.$$

2. Find the moment of inertia about the origin, of the circle $x^2+y^2=a^2$.

$$\textit{Ans.}\ 4\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy = \frac{\pi a^4}{2}.$$

3. Find also the area of the preceding circle by Art. 223.

Ans. πa^2.

4. Find by Art. 223 the area between a straight line and a parabola, each of which joins the origin and the point (a, b), the axis of X being the axis of the parabola.

$$\textit{Ans.} \int_0^a \int_{\frac{bx}{a}}^{b\sqrt{\frac{x}{a}}} dx\,dy = \int_0^b \int_{\frac{ay^2}{b^2}}^{\frac{ay}{b}} dy\,dx = \frac{ab}{6}.$$

5. Find the moment of inertia of the preceding area about the origin.

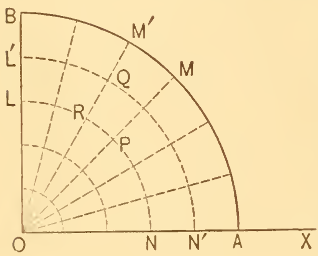
$$Ans.\ \frac{ab}{4}\left(\frac{a^2}{7}+\frac{b^2}{5}\right).$$

6. Find the moment of inertia about the origin, of the area included within the parabola $y^2 = 4ax$, the line $x + y = 3a$, and the axis of X.

$$Ans.\ \int_0^a\int_0^{2\sqrt{ax}}(x^2+y^2)\,dx\,dy+\int_0^{3a}\int_a^{3a-x}(x^2+y^2)\,dx\,dy$$

$$=\int_0^{2a}\int_{\frac{y^2}{4a}}^{3a-y}(x^2+y^2)\,dy\,dx=\frac{314\,a^4}{35}.$$

224. *Double Integration applied to Polar Coördinates.* To find the area of the quadrant of a circle AOB whose radius is a.



In rectangular coördinates, Art. 221, the lines of division consist of two systems, for one of which x is constant, and for the other, y is constant.

So in polar coördinates we have one system of straight lines through the pole, for each of which θ is constant, and another system of circles about the pole as centre, for each of which r is constant.

Let r, θ, which are to be regarded as independent variables, be the coördinates of any point of intersection as P, and $r + dr$, $\theta + d\theta$, the coördinates of Q. Then the area of PQ is $PR \times RQ = rd\theta \cdot dr$.

If we first integrate regarding θ constant, while r varies from 0 to a, we collect all the elements in any sector MOM'.

The second integration sums all the sectors by varying θ from 0 to $\frac{\pi}{2}$.

Hence $$\text{Area } BOA = \int_0^{\frac{\pi}{2}}\int_0^a r\,d\theta\,dr = \frac{\pi a^2}{4}.$$

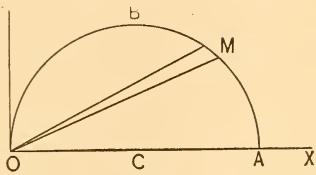
If we reverse the order of integration, integrating first with respect to θ and afterwards with respect to r, we collect all the elements in a circular strip $NLL'N'$, and sum all these strips. This is written

$$\text{Area } BOA = \int_0^a \int_0^{\frac{\pi}{2}} r\,dr\,d\theta.$$

225. If the moment of inertia about O is required, we have for the moment of PQ, $r^2 \cdot r\,d\theta\,dr$. Hence

$$\text{Moment of } BOA = \int_0^{\frac{\pi}{2}} \int_0^a r^3\,d\theta\,dr = \int_0^a \int_0^{\frac{\pi}{2}} r^3\,dr\,d\theta = \frac{\pi a^4}{8}.$$

226. To find by a double integration the area of the semicircle OBA with radius $OC = a$, when polar coördinates are used.

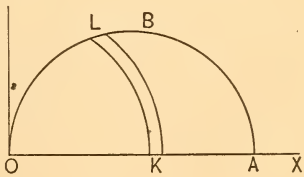


The polar equation of the circle is $r = 2a\cos\theta$. Then, if we integrate first with reference to r, then with reference to θ, we shall have

$$\text{Area } OBA = \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r\,d\theta\,dr = \frac{\pi a^2}{2}.$$

Here, in collecting the elements in a radial strip OM, r varies from 0 to OM. But OM varies with θ, according to the equation of the circle $r = 2a\cos\theta$. Hence the limits are 0 and $2a\cos\theta$.

In collecting all these radial strips for the second integration, θ varies from 0 to $\frac{\pi}{2}$.



By supposing the area composed of concentric circular strips about O as LK, we find

Area OBA

$$= \int_0^{2a} \int_0^{\cos^{-1}\left(\frac{r}{2a}\right)} r\,dr\,d\theta = \frac{\pi a^2}{2}.$$

EXAMPLES.

1. Find the moment of inertia about O of the area of the semicircle in Art. 226. *Ans.* $\dfrac{3\pi a^4}{4}$.

2. Find the moment of inertia about the pole, of the area included by the parabola $r = a\sec^2\dfrac{\theta}{2}$, the initial line OX, and a line at right angles to it through the pole.

$$Ans.\quad \int_0^{\frac{\pi}{2}}\int_0^{a\sec^2\frac{\theta}{2}} r^3\,d\theta\,dr = \frac{48\,a^4}{35}.$$

3. Find the moment of inertia about its centre, of the area of one loop of the lemniscate $r^2 = a^2\cos 2\theta$. *Ans.* $\dfrac{\pi a^4}{16}$.

4. Find by double integration the entire area of the cardioid $r = a\,(1 - \cos\theta)$. *Ans.* $\dfrac{3\pi a^2}{2}$.

5. Find the moment of inertia about the pole, of the area of the preceding cardioid. *Ans.* $\dfrac{35\,\pi a^4}{16}$.